Topology of Sobolev bundles(Variational Problems and Related Topics)

数理解析研究所講究録
1528 巻 2007 年 104-116

Topology of Sobolev bundles

Takeshi Isobe
磯部健志（東京工業大学大学院理工学研究科）
Department of Mathematics
Graduate School of Science and Engineering
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152-8551, JAPAN

1 Introduction

In this notes, we review a recent development obtained by the author concerning the interrelation between topological and analytical properties of Sobolev bundles and its applications to variational problems on principal bundles. Our presentation given here is based on [5].

Let M be an m-dimensional compact Riemannian manifold and G a compact Lie group with bi-invariant metric. There is a faithful unitary representation of G on $\mathbb{R}^l$, i.e., an injection $G \to O(\mathbb{R}^l) = O(l) \subset \mathbb{R}^{l^2}$ for some $l \in \mathbb{Z}$. Thus we may assume without loss of generality that G is a subgroup of $O(l)$. For any open subset $U \subset M$, $k \in \mathbb{Z}$ with $k \geq 1$ and $1 \leq p \leq \infty$, the Sobolev space of functions in U of class $W^{k,p}$, denoted $W^{k,p}(U)$, is defined as the space of all L^p-integrable functions in U whose partial derivatives (in the sense of distributions) of order up to k are also L^p-integrable in U. $W^{k,p}(U,G)$ is then defined as

$$W^{k,p}(U,G) = \{g \in W^{k,p}(U,\mathbb{R}^{l^2}) : g(x) \in G \text{ for a.e. } x \in U\}.$$

By the Sobolev embedding theorem, if k and p satisfy $kp > m$, then $W^{k,p}(U,G) \subset C^0(\overline{U},G)$ and pointwise multiplication and inversion define a continuous multiplication $W^{k,p}(U,G) \times W^{k,p}(U,G) \ni (f,g) \mapsto f \cdot g \in W^{k,p}(U,G)$ and a inversion $W^{k,p}(U,G) \ni f \mapsto f^{-1} \in W^{k,p}(U,G)$. With these, $W^{k,p}(U,G)$ becomes a Banach Lie group. This is not a case for $kp \leq m$ since $W^{k,p}(U,G)$ is not embedded in $C^0(\overline{U},G)$. However, since G is compact, we have $W^{k,p}(U,G) \subset L^\infty(U,\mathbb{R}^{l^2})$ and by the Gagliardo-Nirenberg inequality one can prove that the pointwise multiplication and the inversion operators are defined in $W^{k,p}(U,G)$ and they are continuous. So $W^{k,p}(U,G)$ becomes a topological group even for the case $kp \leq m$.

Recall that the ordinary (i.e., continuous or smooth) principal G-bundle P on M is defined by the following data (see [3], [4]): a) $\{U_\alpha\}_{\alpha\in I}$; an open covering of M, and b) G-equivariant (with respect to the right actions of G) trivializations $P|_{U_\alpha} \xrightarrow[\varphi_\alpha]{\sim} U_\alpha \times G$ over U_α inducing the identity on U_α. These data define a family of transition (or gluing) functions $\{g_{\alpha,\beta}\}_{\alpha,\beta\in I}$ via the formula $\varphi_\alpha \circ \varphi_\beta^{-1}(x,g) = (x, g_{\alpha,\beta}(x)g)$ for $(x,g) \in (U_\alpha \cap U_\beta) \times G$. These satisfy $g_{\alpha,\beta} \in C^0(U_{\alpha,\beta}, G)$ ($C^\infty(U_{\alpha,\beta}, G)$ when considering principal G-bundles of class C^∞), the cocycle condition $g_{\alpha,\beta} \cdot g_{\beta,\gamma} = g_{\alpha,\gamma}$ in $U_{\alpha,\beta,\gamma}$ and $g_{\alpha\alpha} = \mathbf{1}$ in U_α, where we denote $U_{\alpha,\beta,\gamma\dots} = U_\alpha \cap U_\beta \cap U_\gamma \cap \cdots$ and $\mathbf{1}$ is the identity element of G. Conversely from $\{U_\alpha\}_{\alpha\in I}$ and $\{g_{\alpha,\beta}\}_{\alpha,\beta\in I}$ satisfying the cocycle condition, we obtain a principal G-bundle P with trivializations $P|_{U_\alpha} \simeq U_\alpha \times G$ over U_α with transition functions $\{g_{\alpha,\beta}\}$ by gluing the trivial bundles $U_\alpha \times G \to U_\alpha$ via $g_{\alpha,\beta}$ over $U_{\alpha,\beta}$. Thus two different defintions are equivalent. Moreover, two principal G-bundles are equivalent if and only if the associated cocycles are cohomologous, see [4]. Thus the set of equivalence classes of principal G-bundles is described by the Čech cohomology $\check{H}^1(M, \mathcal{C}^0_G)$ (or $\check{H}^1(M, \mathcal{C}^\infty_G)$), where $\mathcal{C}^0_G$ (respectively $\mathcal{C}^\infty_G$) denotes a presheaf defined by $\mathcal{C}^0_G(U) = C^0(U,G)$ (respectively $\mathcal{C}^\infty_G(U) = C^\infty(U,G)$) for any open set $U \subset M$.

Generalizing the Čech cocycle description of smooth G-bundles, the classes of Sobolev G-bundles are defined as follows:

Definition 1.1 *Let $k \in \mathbb{N}$ and $1 \le p \le \infty$. A Sobolev principal G-bundle of class $W^{k,p}$ defined over M is defined by the following two data:*

(1) $\{U_\alpha\}_{\alpha\in I}$; an open covering of M.

(2) $\{g_{\alpha,\beta}\}_{\alpha,\beta\in I}$; a family of G-valued measurable functions satisfying

(a) $g_{\alpha,\beta} \in W^{k,p}(U_{\alpha,\beta}, G)$ for all $\alpha, \beta \in I$ whenever $U_{\alpha,\beta} \neq \emptyset$,

(b) $g_{\alpha,\beta}(x) g_{\beta,\gamma}(x) = g_{\alpha,\gamma}(x)$ for a.e. $x \in U_{\alpha,\beta,\gamma}$ whenever $U_{\alpha,\beta,\gamma} \neq \emptyset$ and $g_{\alpha,\alpha}(x) = \mathbf{1}$ for a.e $x \in U_\alpha$.

We denote such a Sobolev bundle by $P = \langle \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha,\beta}\}_{\alpha,\beta\in I}\rangle$. The set of all Sobolev principal G-bundles of class $W^{k,p}$ is denoted by $\mathcal{P}^{k,p}_G(M)$.

As we noticed before, by the Gagliard-Nirenberg inequality $W^{k,p}(U_{\alpha,\beta,\gamma}, G)$ becomes a group under pointwise multiplication and inversion whenever $U_{\alpha,\beta,\gamma} \neq \emptyset$ so the condition (2-b) in the above definition makes sense.

The bundle isomorphisms for Sobolev principal G-bundles are defined similarly. Namely, we define two Sobolev bundles $P = \langle \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha,\beta}\}_{\alpha,\beta\in I}\rangle$ and $Q = \langle \{V_j\}_{j\in J}, \{h_{jk}\}_{j,k\in J}\rangle$

in the class $\mathcal{P}_G^{k,p}(M)$ are $W^{k,p}$-isomorphic if and only if they define the same cohomology class in $\check{H}^1(M, \mathcal{W}_G^{k,p})$, where $\mathcal{W}_G^{k,p}$ is a presheaf defined by $\mathcal{W}_G^{k,p}(U) = W^{k,p}(U, G)$ for any open set $U \subset M$. That is to say, there exists a refinement $\{W_s\}_{s \in S}$ of both of the open coverings $\{U_\alpha\}$ and $\{V_j\}$ such that $W_s \subset U_{\varphi(s)}$ and $W_s \subset V_{\psi(s)}$, where $\varphi : S \to I$ and $\psi : S \to J$ are refinement maps and a family of maps $\{\rho_s\}_{s \in S}$ such that $\rho_s \in W^{k,p}(W_s, G)$ and $g_{\varphi(s)\varphi(t)} = \rho_s \cdot h_{\psi(s)\psi(t)} \cdot \rho_t^{-1}$ holds in W_{st}.

Sobolev bundles naturally arise as limits of smooth bundles. For example, recall that the moduli space of Yang-Mills connections with equi-bounded Yang-Mills energies on a G-bundle over a manifold with dimension greater than 3 is not compact in general, see [11], [2], [10]. However, any sequence of such connections has a subsequence which *weakly* converges to a *generalized* connection on some *generalized* bundle defined over M. The bundle obtained in such a way has a large set of singularities in general and they in fact belong to a certain kind of Sobolev bundles. Another examples arise naturally in the calculus of variations. For example, consider the problem of minimizing Yang-Mills functional defined over a (smooth) G-bundle. In general, a minimizing sequnece does not converge strongly enough to preserve the smoothness and the topology of the bundle, see [11], [9]. However, for some cases one obtains a *weak* limit of it (it consists of a pair of a *weak* connection and a *weak* G-bundle) and it belongs to a certain kind of Sobolev class.

From the above examples, it is also natural to introduce connections on Sobolev bundles. Recall that a smooth connection defined on a smooth principal G-bundle P (defined by a data $\langle \{U_\alpha\}_{\alpha \in I}, \{g_{\alpha,\beta}\}_{\alpha,\beta \in I} \rangle$) is defined by a family of $\mathfrak{g}$-valued 1-forms $\{A_\alpha\}_{\alpha \in I}$, $A_\alpha \in C^\infty(U_\alpha, T^*U_\alpha \otimes \mathfrak{g})$ satisfying the gluing relation

$$A_\beta = g_{\alpha,\beta}^{-1} d g_{\alpha,\beta} + g_{\alpha,\beta}^{-1} A_\alpha g_{\alpha,\beta} \quad \text{on } U_{\alpha,\beta}. \tag{1.1}$$

For a Sobolev bundle of class $W^{k,p}$, one may also define a connection as a family of $\mathfrak{g}$-valued 1-forms $\{A_\alpha\}_{\alpha \in I}$ belonging to a suitable class of Sobolev space which also satisfies the gluing relation (1.1). In general, a connection loses one more derivatives than the bundle, so one may think that a natural class where A_α lives in is the Sobolev class $W^{k-1,p}$. However, this is not so in general. This is because for $g_{\alpha,\beta} \in W^{k,p}(U_{\alpha,\beta}, G)$ and $A_\alpha \in W^{k-1,p}(U_\alpha, T^*U_\alpha \otimes \mathfrak{g})$ the right hand side of (1.1) does not belong to $W^{k-1,p}$ in general: The Gagliardo-Nirenberg inequality implies $g_{\alpha,\beta}^{-1} d g_{\alpha,\beta} \in W^{k-1,p}(U_{\alpha,\beta})$ while the 1-form $g_{\alpha,\beta}^{-1} A_\alpha g_{\alpha,\beta}$ does not belong to $W^{k-1,p}(U_{\alpha,\beta})$ for the case $kp < m$ in general. However, if we require some additional regularity for A_α, we obtain a right defintion. Since we are primarily interested in the cases $k = 1$ and $k = 2$, we only describe these cases in detail. Extensions to the case $k \geq 3$ is straightforward.

Definition 1.2 *Assume $k = 1$ or $k = 2$ and $1 \leq p \leq \infty$. Let $P = \langle \{U_\alpha\}_{\alpha \in I}, \{g_{\alpha,\beta}\}_{\alpha,\beta \in I} \rangle \in \mathcal{P}_G^{k,p}(M)$. We define the spaces of Sobolev connections $\mathcal{A}^{k-1,p}(P)$ and $\mathfrak{A}^{1,m/2}(P)$ on P as follows:*

(1) The case $k = 1$: $\mathcal{A}^{0,p}(P) = \mathcal{A}^p(P)$ is defined as the set of all $A = \{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \in L^p(U_\alpha, T^*U_\alpha \otimes \mathfrak{g})$ for all $\alpha \in I$ and (1.1) holds a.e. in $U_{\alpha,\beta}$ whenever $U_{\alpha,\beta} \neq \emptyset$.

(2) The case $k = 2$: $\mathcal{A}^{1,p}(P)$ is defined as the set of all $A = \{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \in W^{1,p}(U_\alpha, T^*U_\alpha \otimes \mathfrak{g}) \cap L^{2p}(U_\alpha, T^*U_\alpha \otimes \mathfrak{g})$ for all $\alpha \in I$ and (1.1) holds a.e. in $U_{\alpha,\beta}$ whenever $U_{\alpha,\beta} \neq \emptyset$.

(3) The case $k = 1$ for the critical case $p = m$: For the case $k = 1$ and $p = m$, we define $\mathfrak{A}^{1,m/2}(P)$ as the set of all $A = \{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \in L^m(U_\alpha, T^*U_\alpha \otimes \mathfrak{g})$, $dA_\alpha \in L^{m/2}(U_\alpha, \bigwedge^2 T^*U_\alpha \otimes \mathfrak{g})$ and (1.1) holds a.e. in $U_{\alpha,\beta}$ whenever $U_{\alpha,\beta} \neq \emptyset$.

For a Sobolev bundle $P \in \mathcal{P}_G^{k,p}(M)$, we will normally use the space $\mathcal{A}^{k-1,p}(P)$. However, for $W^{1,m}$-bundles, in some places it is useful to use the intermediate space $\mathfrak{A}^{1,m/2}(P)$.

As we remarked above, the Sobolev bundles arise naturally in variety of ways. However, these have not been studied well unless $kp > m$. This is because the Sobolev space $W^{k,p}(U, G)$ ($U \subset M$ an open set) is not embedded in $C^0(U, G)$ for the case $kp \leq m$ and the Sobolev bundle of class $W^{k,p}$ for such cases does not have a topology in the usual sense, so it seems that it is not useful for applications to geometry and topology. For this reason, only the cases $kp > m$ have been used for applications.

In spite of the above mentioned defect, it is also desireable to develop the topological theory of Sobolev bundles for the cases $kp \leq m$ mainly because its possible applicability to recent developments of higher dimensional moduli and variational problems involving gauge fields, see [10], [8]. Our primary interest is summarized in the following question: Can we define topological invariants for Sobolev bundles which are compatible with respect to the Sobolev topology? If so, does it have some useful applications to geometry of manifolds, variational problems, ... etc ? As the first step to approach this problem we considered in [5] the Sobolev bundles with critical indexes, i.e., the Sobolev bundles of class $W^{k,p}$ with $kp = m$. It truns out that the higher dimensional theory, i.e., the cases $kp < m$ crucially depends on this critical case. For the higher dimensional theory, see [6]. Also the critical case includes the important case, namely $W^{2,2}$-bundles over 4-manifolds.

2 Definition and properties of the topology of Sobolev bundles

Our approach to investigate the topological and analytical properties of Sobolev bundles is based on an approximation theorem of Sobolev bundles by smooth ones.

Theorem 2.1 (Approximation by C^0-bundles) *Let $P = \langle \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha,\beta}\}_{\alpha,\beta\in I}\rangle \in \mathcal{P}_G^{k,p}(M)$ with $kp = m$ ($k \in \mathbb{N}$, $p \geq 1$). Then there exists a refinement $\{V_j\}_{j\in J}$ of $\{U_\alpha\}_{\alpha\in I}$ such that $\#J < \infty$ and $\{V_j\}_{j\in J}$ still covers M, and for any $\epsilon > 0$ there exists a principal G-bundle $P_\epsilon = \langle \{V_j\}_{j\in J}, \{g^\epsilon_{j,k}\}_{j,k\in J}\rangle$ of class $C^0 \cap W^{k,p}$ such that (denoting $\phi : J \to I$ the refinement map, i.e., $V_j \subset U_{\phi(j)}$ for $j \in J$)*

$$\|g_{\phi(j),\phi(k)} - g^\epsilon_{j,k}\|_{W^{k,p}(V_{j,k})} < \epsilon$$

for any $j,k \in J$ with $V_j \cap V_k \neq \emptyset$. If $\#I < \infty$, we can take $J = I$ and $\phi(j) = j$ for all $j \in J$.

For the proof of the above theorem, see [5].

The above approximation theorem is the main tool to define a topological invariant of Sobolev bundles. In the following, we describe the main idea for the critical case $kp = m$. In this case, we show that the category of the $W^{k,p}$-isomorphism classes of $W^{k,p}$-bundles is equivalent to the category of the C^∞-isomorphism classes of C^∞-bundles. On the other hand, for the higher dimensional case $kp < m$, the conclusion of the above approximation theorem does not hold in general. For the construction of topological invariants for such a case, see [6].

In the following theorem, $W^{k,p}$-isomorphism classes of $W^{k,p}$-G-bundles and C^∞-isomorphism classes of C^∞-bundles are denoted by $\hat{\mathcal{P}}_G^{k,p}(M)$ and $\hat{\mathcal{P}}_G^\infty(M)$, respectively. Our main result is the following:

Theorem 2.2 *Let $k \in \mathbb{N}$ and $p \geq 1$ satisfy $kp = m$. There exists a one to one correspondence between $\hat{\mathcal{P}}_G^{k,p}(M)$ and $\hat{\mathcal{P}}_G^0(M)$. That is, the category of $W^{k,p}$-isomorphism classes of principal G-bundles of class $W^{k,p}$ is equivalent to the category of C^0 (or C^∞)-isomorphism classes of principal G-bundles of class C^0 (C^∞, respectively). This is also expressed as the isomorphism of cohomologies $H^1(M, \mathcal{W}_G^{k,p}) \simeq H^1(M, \mathcal{C}_G^0)$.*

The idea of the proof of the above theorem is as follows: For small enough $\epsilon > 0$, it can be shown that the isomorphism classes (as C^0-principal G-bundles) of any approximating C^0-principal G-bundles of P in the sense of Theorem 2.1 depends only on P. This shows that one can associate for each Sobolev principal G-bundle of class $W^{k,p}$ ($kp = m$) an isomorphism class of C^0-principal G-bundles, i.e., the class of approximating C^0-principal G-bundles. This association is natural in the sense that when P is also of class C^0, the corresponding class is the C^0-isomorphism class of P. With this definition, it can be shown that $W^{k,p}$-bundle isomorphisms ($kp = m$) preserve the associated C^0-classes of $W^{k,p}$-bundles. Thus for each $W^{k,p}$-isomorphism class of Sobolev principal G-bundles of class $W^{k,p}$, one can associate an isomorphism class of C^0-principal G-bundles which is the trivial correspondence when restricted to principal G-bundles of class $C^0 \cap W^{k,p}$. In fact, we can prove that this gives a one to one correspondence between the isomorphism classes of Sobolev principal G-bundles of class $W^{k,p}$ ($kp = m$) and that of C^0-principal G-bundles. For a detailed proof, see [5].

We draw here some approximations of the approximation theorem 2.1. In applications, we often need to know the stability of the topology of bundles under various Sobolev topologies. It easily follows from Theorem 2.1 and Theorem 2.2 that the convergence of Sobolev connections in the strong $W^{k-1,p}$-topology ($kp = m$) preserves the underlying topology of the bundle of class $W^{k,p}$ as defined above. Namely, we have (we only state the result for the cases $k = 1$ and $k = 2$. The similar result holds for more general cases):

Theorem 2.3 *Let $P = \langle \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha,\beta}\}_{\alpha,\beta\in I}\rangle$ be a principal G-bundle of class $W^{2,m/2}$ (or $W^{1,m}$) over a m-dimensional compact manifold M. Assume there exists on P a sequence of $\mathcal{A}^{1,m/2}$ ($\mathfrak{A}^{1,m/2}$ respectively)-connections $\{A_n\}$, $A_n = \{A_{\alpha,n}\}_{\alpha\in I}$, and $W^{2,m/2}$ ($W^{1,m}$ respectively)-local trivializations of P with respect to which $\{A_n\}$ convergences strongly to a connection A_∞ in $\mathcal{A}^{1,m/2}$ ($\mathfrak{A}^{1,m/2}$ respectively), i.e., there exist $\sigma_{\alpha,n} \in W^{2,m/2}(U_\alpha, G)$ ($\sigma_{\alpha,n} \in W^{1,m}(U_\alpha, G)$ respectively) and $A_{\alpha,\infty} \in W^{1,m/2}(T^*U_\alpha \otimes \mathfrak{g})$ ($A_{\alpha,\infty} \in L^m(T^*U_\alpha \otimes \mathfrak{g})$ such that $dA_{\alpha,\infty} \in L^{m/2}(\bigwedge^2 T^*U_\alpha \otimes \mathfrak{g})$ respectively) such that $\tilde{A}_{\alpha,n} := \sigma_\alpha^* A_{\alpha,n} = \sigma_{\alpha,n}^{-1} d\sigma_{\alpha,n} + \sigma_{\alpha,n}^{-1} A_{\alpha,n}\sigma_{\alpha,n} \to A_{\alpha,\infty}$ in $W^{1,m/2}$ ($\tilde{A}_{\alpha,n} \to A_{\alpha,\infty}$ in L^m and $d\tilde{A}_{\alpha,n} \to dA_{\alpha,\infty}$ in $L^{m/2}$ respectively). Then $A_\infty := \{A_{\alpha,\infty}\}_{\alpha\in I}$ defines a $\mathcal{A}^{1,m/2}$ ($\mathfrak{A}^{1,m/2}$ respectively)-connection on some principal G-bundle P_∞ of class $W^{2,m/2}$ ($W^{1,m}$ respectively) which has the same associated C^0-class, i.e., $[P_\infty]_0 = [P]_0$.*

In calculus of variations, however, usually the weak Sobolev topology is more important than that of the strong one. The appropriate notion of weak convergence in our setting is given by the theorem of Uhlenbeck [11], [9] as follows: Consider a sequence $\{P_n\} \subset \mathcal{P}_G^\infty(M)$ of smooth G-bundles and connections A_n on them, $A_n \in \mathcal{A}^\infty(P_n)$ such that $\sup_{n\geq 1} \int_M |F_{A_n}|^{m/2} \,\mathrm{dvol}_M < +\infty$. Then there exists a finite set $S \subset M$, a $W^{2,m/2}$-bundle P_∞ over $M \setminus S$, a $W^{1,m/2}$-connection A_∞ on P_∞

and a subsequence of (P_n, A_n) (which we still denote by (P_n, A_n)) such that (P_n, A_n) converges weakly to (P_∞, A_∞) in the following sense:

1) There exists an open covering (by arbitrary small geodesic balls) $\{U_\alpha\}_{\alpha\in I}$, $\bigcup_{\alpha\in I} U_\alpha = M\setminus S$.

2) There exist trivializations $\sigma_{\alpha,n}$ of P_n over U_α: $\sigma_{\alpha,n} : P_n|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times G$.

3) There exists a gluing cocycle $\{g_{\alpha,\beta}\}_{\alpha,\beta\in I}$ such that $g_{\alpha,\beta} \in W^{2,m/2}(U_{\alpha,\beta}, G)$ $(\alpha, \beta \in I)$ and it defines P_∞, $P_\infty = \langle \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha,\beta}\}_{\alpha,\beta\in I}\rangle$. Moreover

$$g_{\alpha,\beta;n} := \sigma_{\alpha,n} \circ \sigma_{\beta,n}^{-1} \rightharpoonup g_{\alpha,\beta} \quad \text{weakly in } W^{2,m/2}(U_{\alpha,\beta}, G).$$

4) There exist $A_{\alpha,\infty} \in W^{1,m/2}(T^*U_\alpha \otimes \mathfrak{g})$ for $\alpha \in I$ which define a $W^{1,m/2}$-connection $A_\infty = \{A_{\alpha,\infty}\}_{\alpha\in I}$ on P_∞ such that

$$A_{\alpha,n} := s_{\alpha,n}^* A_n \rightharpoonup A_\alpha \quad \text{weakly in } W^{1,m/2}(U_\alpha, T^*U_\alpha \otimes \mathfrak{g}),$$

where $s_{\alpha,n}(x) := \sigma_{\alpha,n}^{-1}(x, 1)$ is the canonical local section $s_{\alpha,n} : U_\alpha \to P_n$ defined by $\sigma_{\alpha,n}$.

If the above conditions 1)—4) are satisfied, we call (P_∞, A_∞) the *weak Uhlenbeck limit* of (P_n, A_n) and write $(P_n, A_n) \rightharpoonup (P_\infty, A_\infty)$. In such a case, we also say that P_∞ is the weak Uhlenbeck limit bundle of $\{P_n\}$.

Suppose $(P_n, A_n) \rightharpoonup (P_\infty, A_\infty)$. By the removable singularities theorem of Uhlenbeck [12] for general Sobolev bundles and connections, P_∞ and A_∞ extend across S: There exist a principal G-bundle $\tilde{P}_\infty$ of class $W^{2,m/2}$ over M and a $W^{1,m/2}$-connection $\tilde{A}_\infty$ on $\tilde{P}_\infty$ such that the restrictions of $\tilde{P}_\infty$ and $\tilde{A}_\infty$ to $M\setminus S$ are $W^{2,m/2}$-equivalent to P_∞ and A_∞ respectively.

The following lemma shows that one can also associate a well-defined C^0-isomorphism class of G-bundles over M to such a weak limiting bundle.

Lemma 2.1 *Let S be a finite subset of M. Let $P_\infty = \langle \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha,\beta}\}_{\alpha,\beta\in I}\rangle$ be a principal G-bundle of class $W^{2m/2}$ over $M\setminus S$ and $A_\infty = \{A_\alpha\}_{\alpha\in I}$ a $\mathcal{A}^{1,m/2}$-connection on P_∞. Let (P_i, A_i) $(i = 1, 2)$ be pairs of a principal G-bundle of class $W^{2,m/2}$ over M and a $\mathcal{A}^{1,m/2}$-connection on P_i both of which extend (P_∞, A_∞) across S in the above sense. Then P_1 and P_2 have the same associated C^0-isomorphism class, i.e., $[P_1]_0 = [P_2]_0$.*

For the proof of this lemma, see [5].

From Lemma 2.1, we give the following definition.

Definition 2.1 *Let $P_0 \to M$ be a C^∞-principal G-bundle over M. We denote by $\mathcal{P}^{2,m/2}_{G,weak}(M;P_0)$ the set of principal G-bundles of class $W^{2,m/2}$ over M such that $P \in \mathcal{P}^{2,m/2}_{G,weak}(M;P_0)$ if and only if P is an extension over M of the weak Uhlenbeck limit bundle of a sequence of smooth G-bundles $\{P_n\}$ over M, where each P_n is C^∞-isomorphic to P_0 over M.*

It is not generally true that for $P \in \mathcal{P}^{2,m/2}_{G,\mathrm{weak}}(M;P_0)$, P has the same topology as that of P_0, i.e., $[P]_0 = [P_0]_0$ does not hold in general. For example on S^m (with the round metric), $YM_{m/2}$ is conformally invaraint and one can easily check that (by using the conformal dilation of S^m) for any principal G-bundle $P_0 \to S^m$, $\mathcal{P}^{2,m/2}_{G,\mathrm{weak}}(M;P_0)$ contains the trivial product bundle $S^m \times G \to S^m$.

In the study of the topology of weak Uhlenbeck limit bundles, the following notion turns out to be crucial.

Definition 2.2 *Let M be a smooth m-dimensional manifold equipped with a CW-complex structure. Let P be a principal G-bundle of class $W^{k,p}$ (with $kp = m$, $k \geq 1$). We define the $(m-1)$-class of P, denoted $[P]_{m-1}$, as the isomorphism class of a principal G-bundle $P^0|_{M^{m-1}} \to M^{m-1}$, where $P^0 \to M$ is a principal G-bundle of class C^0 whose isomorphism class corresponds to $[P]_{k,p}$ via the equivalence $\hat{\mathcal{P}}^{k,p}_G(M) \simeq \hat{\mathcal{P}}^0_G(M)$ and M^{m-1} is the $(m-1)$-skeleton of M. Notice that for a smooth principal G-bundle $P \to M$, $[P]_{m-1}$ is simply the isomorphism class of $P|_{M^{m-1}} \to M^{m-1}$.*

The following characterization of $\mathcal{P}^{2,m/2}_{G,\mathrm{weak}}(M;P_0)$ has applications to the calculus of variations.

Theorem 2.4 *Let $P_0 \to M$ be a smooth principal G-bundle over M. Then we have the following: If $P \in \mathcal{P}^{2,m/2}_{G,weak}(M;P_0)$, then $[P]_{m-1} = [P_0]_{m-1}$. Conversely, assume $P \in \mathcal{P}^{2,m/2}_G(M)$ satisfies $[P]_{m-1} = [P_0]_{m-1}$, then $P \in \mathcal{P}^{2,m/2}_{G,weak}(M;P_0)$. Moreover for any $P \in \mathcal{P}^{2,m/2}_{G,weak}(M;P_0)$ and $A \in \mathcal{A}^{1,m/2}(P)$, there exist a sequence of smooth principal G-bundles $\{P_n\}$ over M and a sequence of smooth connections $\{A_n\}$ such that A_n is a connection on P_n, P_n are all isomorphic to P_0 over M and $(P_n, A_n) \rightharpoonup (P, A)$.*

3 Applications to the calculus of variations

As one of the most important applications of the result presented in the previous section, we describe here the result concerning to the existence of solutions to variational problems on principal bundles. Here we are interested in conformally invariant Yang-Mills functional $YM_{m/2}(A) = \int_M |F_A|^{m/2} \operatorname{dvol}_M$ defined on $\mathcal{A}^{1,m/2}(P)$ and its critical points, the $m/2$-Yang-Mills connections, where $F_A = dA + A \wedge A$ is the curvature of A and the pointwise norm of $F_A \in C^\infty(M, \bigwedge^2 T^*M \otimes \operatorname{Ad}(P))$ is defined by the metrics on M and G.

From Theorem 2.4, we can prove that $\mathcal{P}^{2,m/2}_{G,\mathrm{weak}}(M; P_0)$ is sequentially weakly closed (for the proof, see [5]) and obtain:

Corollary 3.1 *The following minimization problem has a solution:*

$$\inf\Big\{ \int_M |F_A|^{m/2}\, dvol_M \, : \, A \in \mathcal{A}^{1,m/2}(P),\ P \in \mathcal{P}^{2,m/2}_G(M) \text{ with } [P]_{m-1} = [P_0]_{m-1}\Big\}.$$

As another application of Theorem 2.1, we further refine the above corollary by establishing a topological compactness and the energy quantization for minimizing sequences. Let P be a smooth principal G-bundle over an m-dimensional manifold M. For such P, we define

$$m(P) = \inf\Big\{ \int_M |F_A|^{m/2} \operatorname{dvol}_M : A \in \mathcal{A}^\infty(P)\Big\},$$

where $\mathcal{A}^\infty(P)$ is the set of smooth connections on P. Notice that $m(P)$ depends only on the conformal class of the metric on M and the isomorphism class of P. Notice that by Theorem 2.1, we also have

$$m(P) = \inf\Big\{ \int_M |F_A|^{m/2} \operatorname{dvol}_M : A \in \mathcal{A}^{1,m/2}(Q)\Big\}, \tag{3.1}$$

where Q is any $W^{2,m/2}$-bundle whose corresponding C^0-isomorphism class equals that of P. For a $W^{2,m/2}$-bundle Q, we also denote by $m(Q)$ the right hand side of (3.1). Thus $m(Q)$ depends only on the conformal class of the metric on M and the $W^{2,m/2}$-isomorphism class of Q.

We consider a minimizing sequence of the above problem: $\{A_n\} \subset \mathcal{A}^\infty(P)$ such that $YM_{m/2}(A_n) := \int_M |F_{A_n}|^{m/2} \operatorname{dvol}_M \to m(P)$ as $n \to \infty$. By the weak compactness and the removable singularities theorems of Uhlenbeck (see [11], [12]), $\{A_n\}$ weakly converges to a $W^{1,m/2}$-connection on some G-bundle of class $W^{2,m/2}$. As we have remarked before, this weak limiting bundle is not isomorphic to P in general. There is a phenomenon known as 'bubbling off of instantons over S^m' which is due to the conformal invariance of $YM_{m/2}$, where $YM_{m/2}(A) = \int_M |F_A|^{m/2} \operatorname{dvol}_M$. Behavior of such a sequence is analysed in the work of Uhlenbeck [11] and Sedlacek [9] for the case $m = 4$. Sedlacek proved that a certain invariant of the bundle (for $G = O(n)$ or $SO(n)$, this coincides with the second Stiefel-Whitney class $w_2(P) \in H^2(M; \mathbb{Z}_2)$

and for $G = U(n)$ it is the first Chern class $c_1(P) \in H^2(M;\mathbb{Z})$) is preserved under the weak convergence and proved the existence of a minimizing connection among connections on bundles with a prescribed invariant. His result follows also from Corollary 3.1. Here we give a more precise result and also treat the general case $m \geq 3$.

The following theorem describes a possible way of loss of compactness of minimizing sequences and establishes a topological compactness and the energy quantization for such sequences:

Theorem 3.1 *Let $P \to M$ be a principal G-bundle as above. Also, let $\{A_n\} \subset \mathcal{A}^\infty(P)$ be a minimizing sequence as above. Then there exist a subsequence $\{A_{n_k}\}$ of $\{A_n\}$, a principal G-bundle P^0 of class $W^{2,m/2} \cap C^0$ and a $W^{1,m/2}$-connection A_0 on P^0 such that (P_{n_k}, A_{n_k}) converges weakly to (P^0, A_0) as specified in the beginning of §4 and we have*

1) P^0 and A_0 satisfy

$$[P^0]_{m-1} = [P]_{m-1} \quad \textit{and} \quad m(P^0) = YM_{m/2}(A_0).$$

2) There exist an integer $l \geq 0$, principal G-bundles $P^1 \to S^m, \ldots, P^l \to S^m$ of class $C^0 \cap W^{2,m/2}$ and connections $A_1 \in \mathcal{A}^{1,m/2}(P^1), \ldots, A_l \in \mathcal{A}^{1,m/2}(P^l)$ such that

$$m_0(P^i) = YM_{m/2,0}(A_i),$$

where $m(P^i)$ and $YM_{m/2}$ considered for the special case $M = S^m$ is denoted by $m_0(P^i)$ and $YM_{m/2,0}$ respectively. Thus $YM_{m/2,0}(A_i) = \int_{S^m} |F_{A_i}|^{m/2}\, dvol_{S^m}$ and $m_0(P^i) = \inf\{YM_{m/2,0}(A) : A \in \mathcal{A}^\infty(P^i)\}$ for $1 \leq i \leq l$.

3) $P \to M$ is isomorphic to the connected sum of $P^0 \to M$, $P^1 \to S^m, \ldots, P^{l-1} \to S^m$ and $P^l \to S^m$ (for the definition, see §5):

$$P \simeq P^0 \# P^1 \# \cdots \# P^l.$$

Moreover, we have

$$m(P) = m(P^0) + m_0(P^1) + \cdots + m_0(P^l).$$

Here we remark that the regularity result proved in [7] shows that the connections A_i $(0 \leq i \leq l)$ and bundles P^i $(0 \leq i \leq l)$ are in fact classes of $C^{1,\alpha}$ and $C^{2,\alpha}$ respectively for some $0 < \alpha < 1$.

For the proof of the above theorem, please see [5].

Theorem 3.1 gives the following existence results as corollaries:

Corollary 3.2 *Let M be a compact m-dimensional Riemannian manifold and G a compact Lie group with $\pi_{m-1}(G) = 0$. Then for any principal G-bundle $P \to M$, the following problem has a solution:*

$$m(P) = \inf\left\{YM_{m/2}(A) : A \in \mathcal{A}^{1,m/2}(P)\right\}.$$

Corollary 3.3 *Assume $\pi_{m-1}(G) \neq 0$. There exists a non-trivial G-bundle $P \to S^m$ such that $m_0(P)$ is attained, i.e., there exists $A \in \mathcal{A}^{1,m/2}(P)$ such that $m_0(P) = \int_{S^m} |F_A|^{m/2}\, dvol_{S^m}$.*

Combined with the Bott periodicity for classical groups, we easily deduce an existence result for G classical groups.

Example 1. For any compact Lie group G, we have $\pi_2(G) = 0$. Thus by Corollary 3.2, $m(P)$ is always attained for any G-bundle $P \to M$ with closed 3-dimensional base manifold M.

Example 2. We apply Corollary 3.2 for classical groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ and $Sp(n)$. Denote $\mathbf{O} = \bigcup_{n\geq 1} O(n)$, where we consider $O(n)$ as a subgroup of $O(n+1)$. We define **SO**, **U**, **SU** and **Sp** similary. Then we have the following stability result for homotopy groups ([1, VII-8]):

$$\pi_i(\mathbf{O}) = \pi_i(\mathbf{SO}) = \pi_i(O(n)) \quad \text{for } n \geq i+2,$$

$$\pi_i(\mathbf{U}) = \pi_i(U(n)) \quad \text{for } n \geq [(i+2)/2],$$

$$\pi_i(SU(n)) = \pi_i(U(n)) \quad \text{for } i > 1,$$

$$\pi_i(\mathbf{Sp}) = \pi_i(Sp(n)) \quad \text{for } n \geq [(i+2)/4].$$

By Bott periodicity [4], we have

$$\pi_{i+2}(\mathbf{U}) = \pi_i(\mathbf{U}), \quad \pi_{i+4}(\mathbf{O}) = \pi_i(\mathbf{Sp}), \quad \pi_{i+4}(\mathbf{Sp}) = \pi_i(\mathbf{O})$$

for $i \geq 0$ and

$$\pi_0(\mathbf{U}) = 0, \quad \pi_1(\mathbf{U}) = \mathbb{Z},$$

$$\pi_0(\mathbf{O}) = \mathbb{Z}_2,\ \pi_1(\mathbf{O}) = \mathbb{Z}_2,\ \pi_2(\mathbf{O}) = 0,\ \pi_3(\mathbf{O}) = \mathbb{Z},\ \pi_4(\mathbf{O}) = 0,$$
$$\pi_5(\mathbf{O}) = 0,\ \pi_6(\mathbf{O}) = 0,\ \pi_7(\mathbf{O}) = \mathbb{Z},$$

$$\pi_0(\mathbf{Sp}) = 0,\ \pi_1(\mathbf{Sp}) = 0,\ \pi_2(\mathbf{Sp}) = 0,\ \pi_3(\mathbf{Sp}) = \mathbb{Z},\ \pi_4(\mathbf{Sp}) = \mathbb{Z}_2,$$
$$\pi_5(\mathbf{Sp}) = \mathbb{Z}_2,\ \pi_6(\mathbf{Sp}) = 0,\ \pi_7(\mathbf{Sp}) = \mathbb{Z}.$$

Thus we have

$$\pi_{m-1}(U(n)) = 0 \quad \text{for } n \geq \frac{m+1}{2} \text{ and } m;\text{ odd},$$

$$\pi_{m-1}(SO(n)) = \pi_{m-1}(O(n)) = 0 \quad \text{for } n \geq m+1 \text{ and } m \equiv 3,5,6,7 \pmod 8,$$

$$\pi_{m-1}(Sp(n)) = 0 \quad \text{for } n \geq [m+1/4] \text{ and } m \equiv 1,2,3,7 \pmod 8.$$

Therefore we obtain: For any G-bundle $P \to M$, where M is a closed Riemannian manifold of dimension m, $m(P)$ is attained for the following cases:

1) m is odd and $G = U(n)$ or $G = SU(n)$ with $n \geq \frac{m+1}{2}$.

2) $m \equiv 3,5,6,7 \pmod 8$ and $G = O(n)$ or $G = SO(n)$ with $n \geq m+1$.

3) $m \equiv 1,2,3,7 \pmod 8$ and $G = Sp(n)$ with $n \geq [m+1/4]$.

For more applications and further development of the theory of Sobolev bundles, please consult [5], [6].

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